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Asymptotic stability of traveling waves for viscous scalar conservation laws

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I dedicate this dissertation to my parents

Abstract

There are many research about Viscous scalar conservation law $u_t + f(u)_x = \mu u_{xx}$. In this paper, we study stability and asymptotic stability of travelling solutions for viscous scalar conservation law. The major result shows that if Rankine Hugoniot condition, the generalized shock condition and some assumptions hold, there exist a solution approaches to the travelling solution, satisfy stability and asymptotic stability problems at corresponding rate. The important feature of this paper is to employ an appropriate weight function to show the stability and asymptotic behavior of the viscous shock waves.

Contents

1

Introduction

We consider the Cauchy problem for viscous scalar conservation laws:

$$u_t + f(u)_x = \mu u_{xx}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{1.0.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.0.2}$$

where $\mu > 0$ is called the coefficient of viscosity, $f \in C^2$ and the initial data $u_0(x)$ satisfies

$$u_0(x) \to u_\pm \quad \text{as} \quad x \to \pm\infty. \tag{1.0.3}$$

Let this scalar viscous conservation law admits smooth travelling wave solutions with shock profile

$$u(x,t) = U(\xi) \quad , \quad \xi = x - st, \tag{1.0.4}$$

$$U(\xi) \to u_\pm \quad \text{as} \quad \xi \to \pm\infty. \tag{1.0.5}$$

s satisfies the Rankine Hugoniot condition

$$s(u_+ - u_-) = f(u_+) - f(u_-), \tag{1.0.6}$$

and the generalized shock condition

$$h(u) = -s(u - u_\pm) + f(u) - f(u_\pm) \begin{cases} < 0 & (u_+ < u < u_-), \\ > 0 & (u_- < u < u_+). \end{cases} \tag{1.0.7}$$

The generalized shock condition means

$$f'(u_+) \le s \le f'(u_-). \tag{1.0.8}$$

About travelling waves for viscous scalar conservation law, Il'in and Oleinik [1], Nishihara [2], Kawashima and Matsumura[3] investigated the asymptotic stability of travelling wave for viscous scalar conservation law with convex function f. However, we need to generalize the condition about f, because the convex function f is limited case. So, there were many researches about the non convex function with some assumptions. Matsumura and Kawashima [4], investigated stability of travelling waves with convex and concave function that has a one inflection point. Jones, Gardner and Kapitula[7] investigated stability of travelling wave and decay rate with C^2 function.

Using these results, Nishihara and Matsumura [6] generalized the asymptotic stability of travelling wave with non convex function f. So, our purpose is to survey on their results on stability and decay rate for any C^2 function.

In section 2, we consider the existence of travelling wave using Lipschitz condition. In section 3, we reformulate the viscous scalar conservation law using section 2 results. In section 4, we check the stability of travelling wave for convex flux case using local existence and a priori estimate. In section 5, we check stability of travelling wave for non convex flux case using local existence and a priori estimate. In section 6 and 7, we show asymptotic decay rate of the viscous shock waves for C^2 function f.

To find the a priori estimates for each case, we use the section 3 and combine them with local existence. Then, we show that stability and asymptotic decay rate for each section. In these progresses, we use an Energy method with appropriate weight function.

Notation 1.

1.0.1 We denote the constant $C_{a,b,..}$ depending on $a, b, ..$ by $C_{a,b,..}$ or only by C.

1.0.2 We denote $f(x) \sim g(x)$ as $x \to a$ when $C^{-1}g < f < Cg$ in a neighbourhood of a.

1.0.3 We denote by L^2 space with the norm

$$||f||^2 = \int_R |f(x)|^2 dx.$$

1.0.4 H^l is the sobolev space of l th order with the norm

$$||f||_l^2 = \Sigma_{j=0}^{l} ||(\frac{\partial}{\partial x})^j f(x)||^2.$$

1.0.5 L^2_w is the weighted L^2 space. $f \in L^2_w$ means $w^{1/2} f \in L^2$ with the norm

$$|f|_w^2 = \int_R w(x)|f(x)|^2 dx.$$

1.0.6 If $w(x) = \langle x \rangle^a = (1 + x^2)^{\alpha/2}$, we write $L^2_w = L^2_\alpha$ and $|\cdot|_w = |\cdot|_\alpha$.

1.0.7 If a weighted function is $\langle x \rangle^\alpha w$, we denote by $f \in L^2_{\alpha,w}$ with the norm

$$|f|_{\alpha,w} = (\int_R < x >^{\alpha} w(x)|f(x)|^2 dx)^{1/2}.$$

For example, if $C^{-1} \leq w(x) \leq C$, we know that $L^2 = H^0 = L_0^2 = L_w^2$ with $||\cdot|| = ||\cdot||_0 = |\cdot|_0 \sim |\cdot|_w$ and that $L_{\alpha,w}^2 = L_\alpha^2$ with $|\cdot|_{\alpha,w} \sim |\cdot|_\alpha$.

2

Existence of travelling wave solution

The first thing we need to check is the existence of travelling wave solutions of viscous scalar conservation law. If f is convex, we get the 2 Lemma.

Lemma 2.0.1. *If* (1.0.1) *admits a travelling wave* $U(x - st)$, *satisfies* $U(\pm\infty) = u_\pm$ *and* f *is convex, then* $u_\pm$ *and* s *must satisfy the Rankine Hugoniot condition and the generalized shock condition.*

Proof. Since we admit $u(x, t) = U(x - st) = U(\xi)$, (1.0.1) satisfy $-sU' + f(U)' = \mu U''$. Integrating over $(\pm\infty, \xi)$, we get

$$\mu U' = -sU + f(U) - c = h(U). \tag{2.0.1}$$

Since $U(\pm\infty) = u_\pm$, $U'(\pm\infty) = 0$ and using $\xi \to \pm\infty$ in (2.0.1) we know

$$h(u_\pm) = 0 \text{ and } c = -su_\pm + f(u_\pm). \tag{2.0.2}$$

It equals to the Rankine Hugoniot condition.
The equation (2.0.1) with $h(u_\pm) = 0$ admits a smooth solution $U(\xi)$ satisfying $U(\pm\infty) = u_\pm$ if and only if

$$\begin{aligned} h(u) &< 0, \quad \text{if} \quad u_+ < u_-, \\ h(u) &> 0, \quad \text{if} \quad u_+ > u_-. \end{aligned} \tag{2.0.3}$$

is done. So, we know that the condition of $h(U)$ must be satisfied because f is convex. Since the generalized shock condition is proved that it's equivalent to the condition of $h(U)$, we proved that the Rankine Hugoniot condition and the generalized shock condition are necessary for the existence of a travelling wave $U(x - st)$ satisfying $U(\pm\infty) = u_\pm$.

□

Lemma 2.0.2. *(Existence of travelling solutions with convex function f) Suppose the Rankine Hugoniot condition and the generalized shock condition hold. Then, there exist a travelling wave $U(x-st)$, satisfying $U(\pm\infty) = u_\pm$. The $U(\xi)$ is a monotone function of ξ.*

Proof. We have

$$\text{Differential equation: } U_\xi = h(U,\xi)\ ,\ U(\pm\infty) = u_\pm$$
$$\text{Initial condition: } U(\xi_*) = U_*$$

To prove Lemma 2.0.2, we use Lipschitz condition.
Since $h(U_k(\xi))$ uniformly converge to $h(U.(\xi)) = U.(\xi)$ by the Pichard iteration, the global solution $U(\xi)$ exist, if $h(U)$ is global lipschitz of U.
So, we need to show $h(U)$ is global lipschitz.
Since h is convex, U can't be $U \geq u_-$ and $U \leq u_+$. It means $U \in [u_+, u_-]$.
So, we get $\forall(\xi, U) \in (\mathbb{R}, (u_+, u_-))$

$$|\tfrac{\partial h}{\partial U}| = |h'(U)| = |f'(U) - s| \leq \sup_{U\in[u_+,u_-]} |f'(U)| + s \leq \max(|f'(u_+)|, |f'(u_-)|) + s \leq K.$$

It means $h(U)$ is Lipschitz. So, the global solution $U(\xi)$ exist.

□

By Lemma 2.0.1 and Lemma 2.0.2, we know:
there exists a travelling wave $U(x-st)$ and it satisfies $U(\pm\infty) = u_\pm$ if and only if $u_\pm$ and shock speed s satisfy the Rankine-Hugoniot condition and the generalized shock condition.

However, we can't use Lemma 2.0.2 for the non-convex function f. So, to prove existence of travelling wave with non convex function f, we need some condition of $h(U)$.

Lemma 2.0.3. *(Existence of travelling wave with $f \in C^2$) Assume Rankine-Hugoniot condition, the generalized shock condition and*

$$|h(U)| \sim |U - u_\pm|^{1+k_\pm}\ ,\ \textit{as } U \to u_\pm, \tag{2.0.4}$$

with $k_\pm \geq 0$. Then there exists a travelling wave solution $U(\xi)$ of viscous scalar conservation law with $U(\pm\infty) = u_\pm$. Also,

$$|U(\xi) - u_\pm| \sim e^{-c|\xi|}\ ,\ \textit{if } f'(u_+) < s < f'(u_-), \tag{2.0.5}$$

$$|U(\xi) - u_\pm| \sim |\xi|^{-1/k_+}\ ,\ \textit{if } s = f'(u_+), \tag{2.0.6}$$

$$|U(\xi)-u_{\pm}| \sim |\xi|^{-1/k_{-}} \ , \ if \ s=f'(u_{-}), \tag{2.0.7}$$

is done as $\xi \to \pm\infty$, where $k_{\pm}$ is denoted to $h^{(n)}(u_{\pm})=0$ if $1 \le n \le k_{\pm}$ and $h^{(n+1)}(u_{\pm}) \neq 0$.

Proof. We have

$$\text{Differential equation: } U_{\xi}=h(U,\xi) \ , \ U(\pm\infty)=u_{\pm}$$
$$\text{Initial condition: } U(\xi_{*})=U_{*}$$

To prove Lemma 2.0.3, we use Lipschitz condition.

Since $h(U_k(\xi))$ uniformly converge to $h(U.(\xi))=U.(\xi)$ by the Pichard iteration, the global solution $U(\xi)$ exist, if $h(U)$ is global lipschitz of U. So, we need to show $h(U)$ is global lipschitz. Since $h<0$ is non convex, $|h(U)| \sim |U-u_{\pm}|^{1+k_{\pm}}$ help $U \to u_{\pm}$ as $\xi \to \infty$. It means U is bounded.

So, we get $\forall(\xi, U) \in (\mathbb{R},(u_{+},u_{-}))$

$$|\tfrac{\partial h}{\partial U}|=|h'(U)|=|f'(U)-s| \le \sup_{U\in[u_{+},u_{-}]}|f'(U)|+s \le \max(|f'(u_{+})|,|f'(u_{-})|)+s \le K.$$

It means $h(U)$ is Lipschitz. So, the global solution $U(\xi)$ exist.

By the implicit formula, we get

1. $f'(u_{+})<s<f'(u_{-})$ case,

Since $h'(u_{\pm})=f'(U)-s \neq 0$, $|h(U)| \sim |U-u_{\pm}|$ as $U \to u_{\pm}$. Therefore, we get

$$\int_{\frac{u_{+}+u_{-}}{2}}^{U(\xi)} \frac{1}{|u-u_{\pm}|}du = ln(u-u_{\pm})|_{u=\frac{u_{+}+u_{-}}{2}}^{u=U(\xi)} = \xi + c.$$

So, we know $|U(\xi)-u_{\pm}| \sim e^{-c|\xi|}$ as $\xi \to \pm\infty$ for some constant c.

2. $s=f'(u_{+})$ or $f'(u_{-})$ case,

In these cases, $|h(U)| \sim |U-u_{\pm}|^{1+k_{\pm}}$ as $U \to u_{\pm}$. Therefore,

$$\int_{\frac{u_{+}+u_{-}}{2}}^{U(\xi)} \frac{1}{|u-u_{\pm}|^{1+k_{\pm}}}du = \frac{c}{(u-u_{\pm})^{k_{\pm}}}|_{u=\frac{u_{+}+u_{-}}{2}}^{u=U(\xi)} = \xi + c.$$

So, we know

$$|U(\xi)-u_{+}| \sim \frac{c}{|\xi|^{1/k_{+}}} \text{ as } \xi \to \infty \text{ for some constant } c \text{ if } s=f'(u_{+}),$$
$$|U(\xi)-u_{-}| \sim \frac{c}{|\xi|^{1/k_{-}}} \text{ as } \xi \to -\infty \text{ for some constant } c \text{ if } s=f'(u_{-}).$$

□

3

Reformulation of the Problem

For the stability and asymptotic stability of travelling wave solution U, we need a condition about U and the initial data u_0. So, we add the assumption $u_0 - U$ is integrable. Since $U(\pm\infty) = u_\pm$, we get

$$\int_R (u_0(x) - U(\pm\infty))dx = \int_R (u_0(x) - u_\pm)dx \gtreqless 0.$$

It means we determine U which satisfies

$$\int_R (u_0(x) - U(x))dx = 0, \tag{3.0.1}$$

and define

$$\psi_0(x) := \int_{-\infty}^{x} (u_0(y) - U(y))dy.$$

Let $U(\xi)$ be the travelling wave solution where satisfy Lemma 2.0.3 and (3.0.1) , then we get

$$u(x,t) = U(\xi) + \psi_\xi(\xi,t) \quad , \text{ where } \xi = x - st. \tag{3.0.2}$$

It means (1.0.1) with the initial data is

$$\psi_t - s\psi_\xi + (f(U + \psi_\xi) - f(U)) = \mu\psi_{\xi\xi}. \tag{3.0.3}$$

$$\psi(\xi, 0) = \int_{-\infty}^{\xi} (u_0 - U)(\eta)d\eta. \tag{3.0.4}$$

(3.0.3) is equivalent to the following form

$$\psi_t + h'(U)\psi_\xi - \mu\psi_{\xi\xi} = F, \tag{3.0.5}$$

$$F = -(f(U + \psi_\xi) - f(U) - f'(U)\psi_\xi). \tag{3.0.6}$$

Now, let select the weight as

$$w = w(U) = \left|\frac{(U-u_+)(U-u_-)}{h(U)}\right|. \tag{3.0.7}$$

Since $|h(U)| \sim |U-u_\pm|^{1+k\pm}$ in Lemma 2.0.3, we get the following form

$$\text{if } f'(u_+) < s < f'(u_-)\ ,\ w(U) \sim C \text{ as } U \to u_\pm \text{ and } L^2_{\alpha,w(U)} = L^2_\alpha, \tag{3.0.8}$$

$$\text{if } f'(u_+) = s < f'(u_-)\ ,\ w(U) \sim |U-u_+|^{-k_+} \text{ as } U \to u_+, \tag{3.0.9}$$

$$\text{or } w(U(\xi)) \sim \langle\xi\rangle \text{ as } \xi \to \infty \text{ and hence } L^2_{w(U)} = L^2_{\langle\xi\rangle_+},$$

$$\text{if } f'(u_+) < s = f'(u_-)\ ,\ w(U) \sim |U-u_-|^{-k_-} \text{ as } U \to u_-, \tag{3.0.10}$$

$$\text{or } w(U(\xi)) \sim \langle\xi\rangle \text{ as } \xi \to -\infty \text{ and hence } L^2_{w(U)} = L^2_{\langle\xi\rangle_-},$$

$$\text{if } f'(u_+) = s = f'(u_-)\ ,\ w(U) \sim |U-u_\pm|^{-k_\pm} \text{ as } U \to u_\pm, \tag{3.0.11}$$

$$\text{or } w(U(\xi)) \sim \langle\xi\rangle \text{ as } \xi \to \pm\infty \text{ and hence } L^2_{w(U)} = L^2_{\langle\xi\rangle_-} = L^2_1.$$

Remark 3.0.1. The reason why the weight function $w(U)$ is introduced and has the value that form is the key point of this thesis. We consider the C^2 function $f(u)$, it's not always $h''(U) = f''(u) > 0$. It means there are problems just doing estimate without using other method. However, with the weight function $w(U)$, $w(U)h(U)$ is convex function, we can estimate much easily. The details comes out later.

We define the solution space

$$X(0,T) = \{\psi \in C^0(0,T; H^2 \cap L^2_{w(U)}), \psi_\xi \in L^2(0,T; H^2 \cap L^2_{w(U)})\}.$$

Then U be the solution globally in time about stability and asymptotic stability for (1.0.1). To prove it, we suppose there is a local existence result.

Proposition 3.0.1. *(local existence)For any $\epsilon_0 > 0$, there exists a positive constant To depending on ϵ_0 such that if $\psi_0 \in H^2 \cap L^2_{w(U)}$ and $||\psi_0||_2 \leq \epsilon_0$, then the problem has a unique solution $\psi \in X(0,T)$ satisfying $||\psi(t)||_2 \leq 2\epsilon_0$ for $0 \leq t \leq T_0$.*

We check the case $u_+ < u_-$, $h(U) \leq 0$ for $U \in [u_+, u_-]$ because the other case, $u_+ > u_-$, $h(U) \geq 0$, is proved in the same method.

4

Stability of travelling wave for convex flux case

In this chapter, we check A priori estimate and Stability of travelling waves for convex flux using continuation argument.

Theorem 4.0.1. *(A priori estimate) Let ψ be a solution of the* (3.0.5) *and satisfying in $X(0,T)$ for a constant $T > 0$. Then there exists a constant $\epsilon_3 > 0$ such that if $\sup_{0 \leq t \leq T} ||\psi(t)||_2 < \epsilon_3$, then $\psi(t)$ satisfies*

$$||\psi(t)||_2^2 + \int_0^t ||\psi_\xi(\tau)||_2^2 d\tau \leq C_2^2(||\psi_0||_2^2),$$

for any $0 \leq t \leq T$.

The key point of this section is to use the standard energy method.

Lemma 4.0.1. *Let $\psi \in X(0,T)$ be a solution of* (3.0.5)*. Then, it satisfies*

$$||\psi(t)|| + \int_0^t \mu ||\psi_\xi(\tau)||^2 d\tau \leq C||\psi_0||^2.$$

Proof. Multiplying 2ψ on (3.0.5) we get

$$2\psi\psi_t + 2\psi\psi_\xi h'(U) - 2\mu\psi\psi_{\xi\xi} = 2\psi F. \tag{4.0.1}$$

It means

$$(\psi^2)_t + (\psi^2 h'(U))_\xi - h''(U)U_\xi\psi^2 - (2\mu\psi\psi_\xi)_\xi + 2\mu\psi_\xi^2 = 2\psi F, \tag{4.0.2}$$

In (4.0.2), $(\cdot)_\xi$ is the term which disappears after integration over R.
Integrating (4.0.2) over $(0,t) \times R$ we get

$$||\psi(t)||^2 + \int_0^t \int_R -(h(U))''U_\xi\psi^2 + 2\mu\psi_\xi^2 d\xi d\tau = ||\psi_0|| + \int_0^t \int_R 2\psi F d\xi d\tau.$$

Since $U_\xi < 0$ and $h''(U) = f''(U) \geq 0$ such that we get

$$||\psi(t)||^2 + \int_0^t 2\mu||\psi_\xi||^2 d\tau \leq ||\psi_0||^2 + \int_0^t \int_R 2\psi F d\xi d\tau.$$

Let

$$N(t) = \sup_{0\leq\tau\leq t} ||\psi(\tau)||_2,$$

and assume $N(t) \leq \epsilon_0$.

By the taylor extension, $|F| = |f(U + \psi_\xi) - f(U) - f'(U)\psi_\xi| = O(\psi_\xi^2)$. It satisfies $|F| \to 0$ as $\xi \to \pm\infty$.

Then, for some ϵ_3 satisfying $N(t) < \epsilon_3 < \epsilon_0$,

$$||\psi(t)||^2 + \int_0^t ||\psi_\xi||^2 d\tau \leq C_2^2||\psi_0||^2.$$

□

Moreover, we need to check the similarly case when we apply ∂_ξ and $\partial_{\xi\xi}$ to rewritten form. Then, we can get a new lemma.

Lemma 4.0.2. *There is a positive constant ϵ_3 where is smaller than ϵ_0 such that if $N(t) \leq \epsilon_3 < \epsilon_0$,*

$$||\psi_\xi(t)||_1^2 + \int_0^t ||\psi_{\xi\xi}(\tau)||_1^2 d\tau \leq C(||(\psi_0)_\xi||_1^2).$$

Proof. For higher order estimate, we apply ∂_ξ to the (3.0.5). Then,

$$\psi_{\xi t} + \{(h(U))'\psi_\xi\}_\xi - \mu\psi_{\xi\xi\xi} = F_\xi.$$

Multiplying $2\psi_\xi$ on (3.0.5) we get

$$(\psi_\xi^2)_t + \{(h(U))'\psi_\xi^2\}_\xi - \{2\mu\psi_{\xi\xi\xi}\psi_\xi\}_\xi + h''(U)U_\xi\psi_\xi^2 + 2\mu\psi_{\xi\xi}^2 = 2\psi_\xi F_\xi. \tag{4.0.3}$$

Integrating (4.0.3) over $(0, t \times R)$ we get

$$||\psi_\xi(t)||^2 + \int_0^t \int_R h''(U)U_\xi\psi_\xi^2 d\xi + 2\mu||\psi_{\xi\xi}(\tau)||d\tau = ||\psi_\xi(0)||^2 + \int_0^t \int_R 2F_\xi\psi_\xi d\xi d\tau.$$

Then, for some ϵ_3 satisfying $N(t) < \epsilon_3 < \epsilon_0$, there exist C such that

$$||\psi_\xi(t)||^2 + \int_0^t \int_R h''(U)U_\xi\psi_\xi^2 d\xi + \mu||\psi_{\xi\xi}(\tau)||d\tau \leq C(||\psi_\xi(0)||^2).$$

By, $-\int_0^t \int_R h''(U)U_\xi\psi_\xi^2 d\xi \geq 0$, we have

$$\begin{aligned} ||\psi_\xi(t)||^2 + \int_0^t \mu||\psi_{\xi\xi}(\tau)||d\tau &\leq C(||\psi_\xi(0)||^2) - \int_0^t \int_R h''(U)U_\xi\psi_\xi^2 d\xi d\tau, \\ &\leq C(||\psi_\xi(0)||^2) + c\int_0^t \int_R \psi_\xi^2 d\xi d\tau, \end{aligned}$$

where $c = \sup_{\xi \in R}\{|h''(U)U_\xi|\} < \infty$.

Applying $\partial_{\xi\xi}$ to (3.0.5) we get

$$\psi_{\xi\xi t} + \{(h(U))'\psi_\xi\}_{\xi\xi} - \mu\psi_{\xi\xi\xi\xi} = F_{\xi\xi},$$

Multiplying $2\psi_{\xi\xi}$ we get

$$2\psi_{\xi\xi}\psi_{\xi\xi t} + \{(h(U))'\psi_\xi\}_{\xi\xi}2\psi_{\xi\xi} - \mu 2\psi_{\xi\xi}\psi_{\xi\xi\xi\xi} = F_{\xi\xi}2\psi_{\xi\xi} \tag{4.0.4}$$

we changed the second term in (4.0.4) to the other form as follows

$$\begin{aligned}
(h'(U)\psi_\xi)_{\xi\xi}\psi_{\xi\xi} &= \{(h'(U)\psi_\xi)_\xi\psi_{\xi\xi}\}_\xi - (h'(U)\psi_\xi)_\xi\psi_{\xi\xi\xi} \\
&= \{(h'(U))_\xi\psi_\xi\psi_{\xi\xi} + h'(U)\psi_{\xi\xi}^2\}_\xi - (h'(U))_\xi\psi_\xi\psi_{\xi\xi\xi} - h'(U)\psi_{\xi\xi}\psi_{\xi\xi\xi} \\
&= \{(h'(U))_\xi\psi_\xi\psi_{\xi\xi} + h'(U)\psi_{\xi\xi}^2\}_\xi - \{\tfrac{1}{2}h'(U)\psi_{\xi\xi}^2\}_\xi + \tfrac{1}{2}(h'(U))_\xi\psi_{\xi\xi}^2 \\
&\quad -\{((h'(U))_\xi\psi_\xi)\psi_{\xi\xi}\}_\xi + ((h'(U))_\xi\psi_\xi)_\xi\psi_{\xi\xi} \\
&= \tfrac{1}{2}\{h'(U)\psi_{\xi\xi}^2\}_\xi + \tfrac{3}{2}(h'(U))_\xi\psi_{\xi\xi}^2 + (h'(U))_{\xi\xi}\psi_\xi\psi_{\xi\xi}.
\end{aligned}$$

It means

$$(\psi_{\xi\xi}^2)_t + \{(h(U))'\psi_{\xi\xi}^2\}_\xi - \{2\mu\psi_{\xi\xi\xi}\psi_{\xi\xi}\}_\xi + 3h'(U)_\xi\psi_{\xi\xi}^2 + 2h'(U)_{\xi\xi}\psi_\xi\psi_{\xi\xi} + 2\mu\psi_{\xi\xi\xi}^2 = 2\psi_{\xi\xi}F_{\xi\xi}. \tag{4.0.5}$$

Integrating (4.0.5) over $(0,t)\times R$ we get

$$\begin{gathered}
||\psi_{\xi\xi}(t)||^2 + \int_0^t\int_R +3h'(U)_\xi\psi_{\xi\xi}^2 + 2h'(U)_{\xi\xi}\psi_\xi\psi_{\xi\xi}d\xi + 2\mu||\psi_{\xi\xi\xi}(\tau)||d\tau = \\
||\psi_{\xi\xi}(0)||^2 + \int_0^t\int_R 2F_{\xi\xi}\psi_{\xi\xi}d\xi d\tau.
\end{gathered}$$

Then, for some ϵ_3 satisfying $N(t) < \epsilon_3 < \epsilon_0$, there exist C such that

$$||\psi_{\xi\xi}(t)||^2 + \int_0^t\int_R \mu\psi_{\xi\xi\xi}^2(\tau)d\xi d\tau \le C(||\psi_{\xi\xi}(0)||^2) - \int_0^t\int_R 3h'(U)_\xi\psi_{\xi\xi}^2 + 2h'(U)_{\xi\xi}\psi_\xi\psi_{\xi\xi}d\xi d\tau.$$

Since $-\int_0^t\int_R 3h'(U)_\xi\psi_{\xi\xi}^2 + 2h'(U)_{\xi\xi}\psi_\xi\psi_{\xi\xi}d\xi d\tau \le c\int_0^t\int_R \psi_\xi^2 + \psi_{\xi\xi}^2 d\xi d\tau$,
where $c = 5\sup_{\xi\in R}\{|(h'(U))_\xi + (h'(U))_{\xi\xi}|\} < \infty$, we have

$$||\psi_{\xi\xi}(t)||^2 + \int_0^t\int_R \mu\psi_{\xi\xi\xi}^2(\tau)d\xi d\tau \le C(||\psi_{\xi\xi}(0)||^2) + c\int_0^t ||\psi_{\xi\xi}(\tau)||^2 + ||\psi_\xi(\tau)||^2 d\tau.$$

Combining with these higher order cases, we get

$$\begin{gathered}
||\psi_\xi(t)||^2 + ||\psi_{\xi\xi}(t)||^2 + \int_0^t ||\psi_{\xi\xi}(\tau)||^2 + ||\psi_{\xi\xi\xi}(\tau)||^2 d\tau \le C(||(\psi_0)_\xi||^2 + ||(\psi_0)_{\xi\xi}||^2) \\
+c\int_0^t ||\psi_{\xi\xi}(\tau)||^2 + ||\psi_\xi(\tau)||^2 d\tau.
\end{gathered}$$

□

Combining Lemma 4.0.1 and Lemma 4.0.2. we get

$$||\psi(t)||_2^2 + \int_0^t ||\psi_\xi(\tau)||_2^2 d\tau \leq C(||\psi_0||_2^2).$$

where constant C dependent of $N(t)$. So, the proof of A priori estimate is done.
Let check the Stability of travelling wave.

Theorem 4.0.2. *(Stability) Suppose $\psi_0 \in H^2$. Then there exists a positive constant ϵ_2 and C_2 such that if $||\psi_0||_2 < \epsilon_2$, the problem has a unique global solution $\psi \in X(0, \infty)$,*

$$||\psi(t)||_2^2 + \int_0^t ||\psi_\xi(\tau)||_2^2 d\tau \leq C_2^2(||\psi_0||_2^2). \tag{4.0.6}$$

for any $t \geq 0$. Also, ψ_ξ tends to 0 in the L^∞ norm as $t \to \infty$,

$$sup_{\xi \in R} |\psi_\xi(\xi, t)| \to 0 \text{ , as } t \to \infty.$$

Proof. To prove this theorem, we have to use Continuation argument.
Suppose $\epsilon_2 = \min\{\epsilon_3/2, \epsilon_3/2C_3\}$, $C_2 = C_3$.
By the local existence, we put $||\psi_0||_2^2 \leq \epsilon_2^2 \leq \epsilon_3^2/4$. Because of T_0 is dependent of ϵ_0 , T_0 has the value $T_0(\epsilon_3)$ such that the solution exists on $[0, T_0(\epsilon_3)]$. It means

$$||\psi(t)||_2^2 \leq 4(||\psi_0||_2^2) \leq \epsilon_3^2, \text{ for } t \in [0, T_0].$$

By A priori estimate with $T = T_0(\epsilon_3)$, we know

$$||\psi(t)||_2^2 + \int_0^t ||\psi_\xi(\tau)||_2^2 d\tau \leq C_3^2(||\psi_0||_2^2),$$

for $0 \leq t \leq T$. So we get

$$||\psi(t)||_2^2 \leq C_3^2(||\psi_0||_2^2) \leq C_3^2 \epsilon_2^2 \leq \frac{\epsilon_3^2}{4}.$$

It means

$$||\psi(T_0)||_2^2 \leq C_3^2 \epsilon_2^2 \leq \frac{\epsilon_3^2}{4}.$$

Using above results, we know $\psi(\xi, T_0) \in H^2$ and $||\psi(T_0)||_2^2 \leq ||\psi(T_0)||_2^2 \leq \frac{\epsilon_3^2}{4}$. So, we can apply Local existence again at the start time T_0. Then,

$$||\psi(t)||_2^2 \leq 4(||\psi_0||_2^2) \leq \epsilon_3^2 \text{ for } t \in [T_0, 2T_0].$$

It means Local existence holds for $t \in [0, 2T_0]$.
Also, by A priori estimate, we get

$$||\psi(t)||_2^2 + \int_0^t ||\psi_\xi(\tau)||_2^2 d\tau \leq C_3^2(||\psi_0||_2^2),$$
$$\Rightarrow ||\psi(t)||_2^2 \leq C_3^2(||\psi_0||_2^2) \leq C_3^2 \epsilon_2^2 \leq \frac{\epsilon_3^2}{4}.$$

for $t \in [0, 2T_0]$.

Using the same methods, Local existence and A priori estimate hold for $t \in [0, nT_0], \quad n \in \mathbb{N}$. It means a global solution $\psi \in X(0, \infty)$ exist.

The remain thing is to show ψ_ξ tends to 0 in the L^∞ norm as $t \to \infty$,

$$sup_{\xi \in R} |\psi_\xi(\xi, t)| \to 0 \text{ , as } t \to \infty. \tag{4.0.7}$$

Using the Gagliardo-Nirenburg interpolation inequality, we get

$$\sup_{\xi \in R} |\psi_\xi(t)| \leq ||\psi_{\xi\xi}(t)||^{1/2} ||\psi_\xi||^{1/2}. \tag{4.0.8}$$

Since

$$||\psi(t)||^2 + \int_0^t \mu ||\psi_\xi(\tau)||^2 d\tau \leq C||\psi_0||^2$$
$$||\psi_\xi(t)||^2 + \int_0^t \mu ||\psi_{\xi\xi}(\tau)||^2 d\tau \leq C||\psi_\xi(0)||^2 + c\int_0^t ||\psi_\xi(\tau)||^2 d\tau \leq C||\psi_0||_1^2$$
$$||\psi_{\xi\xi}(t)||^2 + \int_0^t \mu ||\psi_{\xi\xi\xi}(\tau)||^2 d\tau \leq C||\psi_{\xi\xi}(0)||^2 + c\int_0^t ||\psi_\xi(\tau)||^2 + ||\psi_{\xi\xi}(\tau)||^2 d\tau \leq C||\psi_0||_2^2$$

are done, we know $||\psi_\xi(t)||$ and $||\psi_{\xi\xi}(t)||$ are uniformly bounded for $t \geq 0$.

Also, since $||\psi_\xi(t)||^2$ is integrable over t and $\frac{d}{dt}||\psi_\xi(t)||^2$ is also integrable over t and satisfies

$$\int_0^t (\tfrac{d}{dt}||\psi_\xi(\tau)||^2) d\tau = ||\psi_\xi(t)||^2 - ||\psi_\xi(0)||^2 \leq C||\psi_\xi(0)||^2,$$

we know $||\psi_\xi(t)||^2$ is lipschitz continuous. Therefore $||\psi_\xi(t)||^2$ is uniformly continuous. It means

$$||\psi_\xi(t)|| \to 0 \text{ , as } t \to \infty. \tag{4.0.9}$$

So we get,

$$\sup_{\xi \in R} |\psi_\xi(t)| \leq ||\psi_{\xi\xi}(t)||^{1/2} ||\psi_\xi||^{1/2} \to 0, \text{ as } t \to \infty \tag{4.0.10}$$

Therefore, Theorem 4.0.2 is done. □

5

Stability of travelling wave for non convex flux case

In this section, we check A priori estimate and Stability of travelling waves for non convex flux using continuation argument.
The key point of this section is to use the energy method with weighted function $w(U)$.

Theorem 5.0.1. *(A priori estimate) Let ψ be a solution of the* (3.0.5) *and satisfying in $X(0,T)$ for a positive constant T. Then there exists a positive constant ϵ_3 such that if* $\sup_{0\le t\le T}||\psi(t)||_2 < \epsilon_3$*, then $\psi(t)$ satisfies*

$$||\psi(t)||_2^2 + |\psi(t)|_{w(U)}^2 + \int_0^t ||\psi_\xi(\tau)||_2^2 + |\psi_\xi(\tau)|_{w(U)}^2 d\tau \le C_2^2(||\psi_0||_2^2 + |\psi_0|_{w(U)}^2),$$

for any $0 \le t \le T$.

Lemma 5.0.1. *Let $\psi \in X(0,T)$ be a solution of the reformulation form. Then, it satisfies*

$$\frac{1}{2}|\psi(t)|_{w(U)}^2 + \int_0^t ||\sqrt{-U_\xi}\psi(\tau)||^2 + \mu|\psi_\xi(\tau)|_{w(U)}^2 d\tau \le \frac{1}{2}|\psi_0|_{w(U)}^2 + \int_0^t \int_R w(U)\psi F dx d\tau,$$

Proof. Multiplying the (3.0.5) by $2w(U)\psi$ we get

$$2w\psi\psi_t + 2w\psi\psi_\xi h'(U) - 2w\mu\psi\psi_{\xi\xi} = 2w\psi F. \tag{5.0.1}$$

Using the product rule, we get

$$2w\psi\psi_t + 2w\psi\psi_\xi h'(U) + 2w'\psi\psi_\xi h(U) - 2w'\psi\psi_\xi h(U) - 2w\mu\psi\psi_{\xi\xi} = 2w\psi F, \tag{5.0.2}$$

$$\Rightarrow (w\psi^2)_t + 2(wh)'\psi\psi_\xi - 2w'\psi\psi_\xi h(U) - 2w\mu\psi\psi_{\xi\xi} = 2w\psi F, \tag{5.0.3}$$

$$\Rightarrow (w\psi^2)_t + ((wh)'\psi^2)_\xi - (wh)''U_\xi\psi^2 - 2w'\psi\psi_\xi h(U) - 2w\mu\psi\psi_{\xi\xi} = 2w\psi F, \tag{5.0.4}$$

In the chapter 2, we know $h(U) = \mu U_\xi$ such that we get

$$(w\psi^2)_t + ((wh)'\psi^2)_\xi - (wh)''U_\xi\psi^2 - 2\mu w'U_\xi\psi\psi_\xi - 2w\mu\psi\psi_{\xi\xi} = 2w\psi F. \tag{5.0.5}$$

Using the product rule, we get

$$(w\psi^2)_t + ((wh)'\psi^2)_\xi - (wh)''U_\xi\psi^2 - 2(\mu w\psi\psi_\xi)_\xi + 2\mu w\psi_\xi^2 = 2w\psi F. \tag{5.0.6}$$

It means

$$\begin{aligned}(\tfrac{1}{2}w(U)\psi^2)_t + (\tfrac{1}{2}(w(U)h(U))'\psi^2 - \mu w(U)\psi\psi_\xi)_\xi \\ +\mu w(U)\psi_\xi^2 - \tfrac{1}{2}(w(U)h(U))''U_\xi\psi^2 = w(U)\psi F.\end{aligned}$$

Integrating the result over $(0,t)\times R$ we get

$$\tfrac{1}{2}|\psi(t)|^2_{w(U)} + \int_0^t\int_R \mu w(U)\psi_\xi^2 - \tfrac{1}{2}(w(U)h(U))''U_\xi\psi^2 d\xi d\tau = \tfrac{1}{2}|\psi_0|^2_{w(U)} + \int_0^t\int_R w(U)\psi F d\xi d\tau.$$

Since we selected the weight $w = w(U) = |\frac{(U-u_+)(U-u_-)}{h(U)}|$, we know $w(U)h(U)$ is convex function of U such that $\frac{1}{2}(wh)'' = 1$.

$$\tfrac{1}{2}|\psi(t)|^2_{w(U)} + \int_0^t \mu|\psi_\xi(\tau)|^2_{w(U)} + ||-\sqrt{U_\xi}\psi(\tau)||^2 d\tau = \tfrac{1}{2}|\psi_0|^2_{w(U)} + \int_0^t\int_R w(U)\psi F d\xi d\tau$$

□

Let

$$N(t) = \sup_{0\le\tau\le t}||\psi(\tau)||_2$$

and assume $N(t) \le \epsilon_0$.
By the taylor extension, $|F| = |f(U+\psi_\xi) - f(U) - f'(U)\psi_\xi| = O(\psi_\xi^2)$. It satisfies $|F| \to 0$ as $\xi \to \pm\infty$.

Then, for some ϵ_3 satisfying $N(t) < \epsilon_3 < \epsilon_0$,

$$|\psi(t)|^2_{w(U)} + \int_0^t |\psi_\xi(\tau)|^2_{w(U)} d\tau \le C|\psi_0|^2_{w(U)}$$

Moreover, we need to check the similarly case when we apply ∂_ξ and $\partial_{\xi\xi}$ to (3.0.5). Then, we can get a new lemma

Lemma 5.0.2. *There is a positive constant ϵ_3 where is smaller than ϵ_0 such that if $N(t) \le \epsilon_3 < \epsilon_0$,*

$$||\psi_\xi(t)||_1^2 + \int_0^t ||\psi_{\xi\xi}(\tau)||_1^2 d\tau \le C(||(\psi_0)_\xi||_1^2 + |\psi_0|^2_{w(U)})$$

Proof. For the estimates of ψ_ξ and $\psi_{\xi\xi}$, we apply ∂_ξ and $\partial_{\xi\xi}$ to (3.0.5) and multiply ψ and $\psi_{\xi\xi}$. Applying ∂_ξ to (3.0.5), and multiplying $2\psi_\xi$ to the result, we get

$$2\psi_\xi\psi_{\xi t} + 2(h'(U)\psi_\xi)_\xi\psi_\xi - 2\mu\psi_{\xi\xi\xi}\psi_\xi = 2F_\xi\psi_\xi. \tag{5.0.7}$$

Using the product rule, we get

$$\begin{aligned}(\psi_\xi^2)_t + 2h''(U)U_\xi\psi_\xi^2 + \{(h)'\psi_\xi^2\}_\xi - (h)''U_\xi\psi_\xi^2 - \{2\mu\psi_\xi\psi_{\xi\xi}\}_\xi + 2\mu\psi_{\xi\xi}^2 \\ = 2F_\xi\psi_\xi\end{aligned}$$

Integrating over $(0,t)\times\mathbb{R}$, we get

$$||\psi_\xi(t)||^2 - ||\psi_\xi(0)||^2 + \int_0^t\int_R h''U_\xi\psi_\xi^2 d\xi d\tau + \int_0^t 2\mu||\psi_{\xi\xi}(\tau)||^2 d\tau = \int_0^t\int_R 2F_\xi\psi_\xi d\xi d\tau \tag{5.0.8}$$

Under the smallness assumption on $N(T)$, we get

$$||\psi_\xi(t)||^2 + \int_0^t\int_R h''U_\xi\psi_\xi^2 d\xi + 2\mu||\psi_{\xi\xi}(\tau)||^2 d\tau \le ||\psi_\xi(0)||^2 \tag{5.0.9}$$

Since U is a smooth function and both U_ξ and $U_{\xi\xi}$ converge to 0 as $\xi\to\pm\infty$, $|U_\xi|$ and $|U_{\xi\xi}|$ are bounded, so is $\sup_{\xi\in\mathbb{R}}|h''(U)U_\xi| < \infty$. It means

$$||\psi_\xi(t)||^2 + \int_0^t 2\mu||\psi_{\xi\xi}(\tau)||^2 d\tau \le ||\psi_\xi(0)||^2 - \int_0^t\int_R 2h''U_\xi\psi_\xi^2 d\xi d\tau \tag{5.0.10}$$

$$\begin{aligned}\le ||\psi_\xi(0)||^2 + c\textstyle\int_0^t\int_R \psi_\xi^2 d\xi d\tau \le ||\psi_\xi(0)||^2 + c\int_0^t\int_R w\psi_\xi^2 d\xi d\tau \\ \le ||\psi_\xi(0)||^2 + C|\psi(0)|^2_{w(U)}\end{aligned}$$

where $c = 2\sup_{\xi\in\mathbb{R}}|h''(U)U_\xi|$
So, the estimates of ψ_ξ follows

$$||\psi_\xi(t)||^2 + \int_0^t \mu|\psi_{\xi\xi}(\tau)|^2_{w(U)} d\tau \le C(||\psi_\xi(0)||^2 + |\psi(0)|^2_{w(U)}) \tag{5.0.11}$$

Applying $\partial_{\xi\xi}$ to (3.0.5), and multiplying $2\psi_{\xi\xi}$ to the result, we get

$$2\psi_{\xi\xi}\psi_{\xi\xi t} + 2(h'(U)\psi_\xi)_{\xi\xi}\psi_{\xi\xi} - 2\mu\psi_{\xi\xi\xi\xi}\psi_{\xi\xi} = 2F_{\xi\xi}\psi_{\xi\xi}. \tag{5.0.12}$$

Using the product rule, we get

$$2\psi_{\xi\xi}\psi_{\xi\xi t} + 2(h'(U)\psi_\xi)_{\xi\xi}\psi_{\xi\xi} - \{2\mu\psi_{\xi\xi\xi}\psi_{\xi\xi}\}_\xi + 2w\mu\psi_{\xi\xi\xi}^2 = 2wF_{\xi\xi}\psi_{\xi\xi}.$$

The second term, $2(h'(U)\psi_\xi)_{\xi\xi}\psi_{\xi\xi}$ is calculated as follows

$$\begin{aligned}2(h'(U)\psi_\xi)_{\xi\xi}\psi_{\xi\xi} &= \{(2h'\psi_\xi)_\xi\psi_{\xi\xi}\}_\xi - (2h'\psi_\xi)_\xi\psi_{\xi\xi\xi}\\ &= \{2h'\psi_{\xi\xi}^2 + 2h''U_\xi\psi_\xi\psi_{\xi\xi}\}_\xi - 2h''U_\xi\psi_\xi\psi_{\xi\xi\xi} - 2h'\psi_{\xi\xi}\psi_{\xi\xi\xi}\\ &= \{2h'\psi_{\xi\xi}^2 + 2h''U_\xi\psi_\xi\psi_{\xi\xi}\}_\xi + \{-(2h''U_\xi\psi_\xi\psi_{\xi\xi})_\xi + (2h''U_\xi)_\xi\psi_\xi\psi_{\xi\xi} + 2h''U_\xi\psi_{\xi\xi}\}\\ &\quad +\{-(h'\psi_{\xi\xi}^2)_\xi + h''U_\xi\psi_{\xi\xi}^2\}\\ &= \{h'\psi_{\xi\xi}^2\}_\xi + 3h''U_\xi\psi_{\xi\xi}^2 + 2(h')_{\xi\xi}\psi_\xi\psi_{\xi\xi}\end{aligned}$$

So, we get

$$\begin{aligned}&2\psi_{\xi\xi}\psi_{\xi\xi t} + \{h'\psi_{\xi\xi}^2\}_\xi + 3h''U_\xi\psi_{\xi\xi}^2 + 2(h')_{\xi\xi}\psi_\xi\psi_{\xi\xi}\\ &-\{2\mu\psi_{\xi\xi\xi}\psi_{\xi\xi}\}_\xi + 2\mu\psi_{\xi\xi\xi}^2 = 2F_{\xi\xi}\psi_{\xi\xi}.\end{aligned}$$

Integrating over $(0,t)\times\mathbb{R}$, we get

$$\begin{aligned}&||\psi_{\xi\xi}(t)||^2 - ||\psi_{\xi\xi}(0)||^2 + \textstyle\int_0^t\int_R 3(h)''U_\xi\psi_{\xi\xi}^2 + 2(h')_{\xi\xi}\psi_\xi\psi_{\xi\xi}d\xi d\tau\\ &+\textstyle\int_0^t 2\mu||\psi_{\xi\xi\xi}(\tau)||^2 d\tau = \int_0^t\int_R 2F_{\xi\xi}\psi_{\xi\xi}d\xi d\tau.\end{aligned}$$

Under the smallness assumption on $N(T)$, we get

$$||\psi_{\xi\xi}(t)||^2 + \textstyle\int_0^t 2\mu||\psi_{\xi\xi\xi}(\tau)||^2 d\tau \le ||\psi_{\xi\xi}(0)||^2 - \int_0^t\int_R 3(h)''U_\xi\psi_{\xi\xi}^2 + 2(h')_{\xi\xi}\psi_\xi\psi_{\xi\xi}d\xi d\tau.$$

Since U is a smooth function and both U_ξ and $U_{\xi\xi}$ converge to 0 as $\xi\to\pm\infty$, $|U_\xi|$ and $|U_{\xi\xi}|$ are bounded, so is $\sup_{\xi\in\mathbb{R}}|(h'(U))_\xi + (h'(U))_{\xi\xi}| < \infty$. It means

$$\begin{aligned}&||\psi_{\xi\xi}(t)||^2 + \textstyle\int_0^t 2\mu||\psi_{\xi\xi\xi}(\tau)||^2 d\tau \le ||\psi_{\xi\xi}(0)||^2 - \int_0^t\int_R 3(h)''U_\xi\psi_{\xi\xi}^2 + 2(h')_{\xi\xi}\psi_\xi\psi_{\xi\xi}d\xi d\tau\\ &\le ||\psi_{\xi\xi}(0)||^2 + c\textstyle\int_0^t ||\psi_{\xi\xi}(\tau)|| + ||\psi_\xi(\tau)||d\tau \le ||\psi_{\xi\xi}(0)||^2 + C(||\psi_\xi(0)||^2 + |\psi(0)|_{w(U)}^2)\end{aligned}$$

where $c = 5\sup_{\xi\in\mathbb{R}}|(h'(U))_\xi + (h'(U))_{\xi\xi}| < \infty$.
So, the estimates of $\psi_{\xi\xi}$ follows

$$||\psi_{\xi\xi}(t)||^2 + \int_0^t 2\mu||\psi_{\xi\xi\xi}(\tau)||^2 d\tau \le C(||\psi_{\xi\xi}(0)||^2 + ||\psi_\xi(0)||^2 + |\psi(0)|_{w(U)}^2) \tag{5.0.13}$$

Combining (5.0.11) with (5.0.13) , we get

$$||\psi_\xi(t)||_1^2 + \textstyle\int_0^t ||\psi_{\xi\xi}(\tau)||_1^2 d\tau \le C(||\psi_\xi(0)||_1^2 + |\psi_0|_{w(U)}^2).$$

□

So, Combineing Lemma 5.0.1 with Lemma 5.0.2, we get

$$||\psi(t)||_2^2 + |\psi(t)|_{w(U)}^2 + \textstyle\int_0^t ||\psi_\xi(\tau)||_2^2 + |\psi_\xi(\tau)|_{w(U)}^2 d\tau \le C(||\psi_0||_2^2 + |\psi_0|_{w(U)}^2).$$

where constant C dependent of $N(t)$. So, the proof of A priori estimate is done.
Let check the Stability of travelling wave.

Theorem 5.0.2. *(Stability) Suppose $\psi_0 \in H^2 \cap L^2_{w(U)}$. Then there exists a positive constant ϵ_2 and C_2 such that if $||\psi_0||_2 + |\psi_0|_{w(U)} < \epsilon_2$, the problem has a unique global solution $\psi \in X(0,\infty)$,*

$$||\psi(t)||_2^2 + |\psi(t)|_{w(U)}^2 + \int_0^t ||\psi_\xi(\tau)||_2^2 + |\psi_\xi(\tau)|_{w(U)}^2 d\tau \leq C_2^2(||\psi_0||_2^2 + |\psi_0|_{w(U)}^2). \tag{5.0.14}$$

for any $t \geq 0$. Also, ψ_ξ tends to 0 in the L^∞ norm as $t \to \infty$,

$$sup_{\xi \in R}|\psi_\xi(\xi,t)| \to 0 \text{ , as } t \to \infty. \tag{5.0.15}$$

Proof. To prove this theorem, we have to use continuation argument, again. The proof is similar to Theorem 4.0.2. The difference point is the weighted L^2_w norm.
Suppose $\epsilon_2 = \min\{\epsilon_3/2, \epsilon_3/2C_3\}$, $C_2 = C_3$.
By the local existence, we put $||\psi_0||_2^2 + |\psi_0|_{w(U)}^2 \leq \epsilon_2^2 \leq \epsilon_3^2/4$. Because of T_0 is dependent of ϵ_0 , T_0 has the value $T_0(\epsilon_3)$ such that the solution exists on $[0, T_0(\epsilon_3)]$. It means

$$||\psi(t)||_2^2 + |\psi(t)|_{w(U)}^2 \leq 4(||\psi_0||_2^2 + |\psi_0|_{w(U)}^2) \leq \epsilon_3^2 \text{ for } t \in [0, T_0].$$

By A priori estimate with $T = T_0(\epsilon_3)$, we know for $0 \leq t \leq T$

$$||\psi(t)||_2^2 + |\psi(t)|_{w(U)}^2 + \int_0^t ||\psi_\xi(\tau)||_2^2 + |\psi_\xi(\tau)|_{w(U)}^2 d\tau \leq C_3^2(||\psi_0||_2^2 + |\psi_0|_{w(U)}^2).$$

So, we get

$$||\psi(t)||_2^2 + |\psi(t)|_{w(U)}^2 \leq C_3^2(||\psi_0||_2^2 + |\psi_0|_{w(U)}^2) \leq C_3^2\epsilon_2^2 \leq \tfrac{\epsilon_3^2}{4}.$$

It means

$$||\psi(T_0)||_2^2 + |\psi(T_0)|_{w(U)}^2 \leq C_3^2\epsilon_2^2 \leq \tfrac{\epsilon_3^2}{4}.$$

Using above results, we know $\psi(\xi, T_0) \in H^2 \cap L^2_{w(U)}$ and $||\psi(T_0)||_2^2 \leq ||\psi(T_0)||_2^2 + |\psi(T_0)|_{w(U)}^2 \leq \frac{\epsilon_3^2}{4}$. So, we can apply Local existence again at the start time T_0,

$$||\psi(t)||_2^2 + |\psi(t)|_{w(U)}^2 \leq 4(||\psi_0||_2^2 + |\psi_0|_{w(U)}^2) \leq \epsilon_3^2 \text{ for } t \in [T_0, 2T_0].$$

It means Local existence holds for $t \in [0, 2T_0]$.
Also, by A priori estimate,

$$\begin{gathered} ||\psi(t)||_2^2 + |\psi(t)|_{w(U)}^2 + \int_0^t ||\psi_\xi(\tau)||_2^2 + |\psi_\xi(\tau)|_{w(U)}^2 d\tau \leq C_3^2(||\psi_0||_2^2 + |\psi_0|_{w(U)}^2) \\ \Rightarrow ||\psi(t)||_2^2 + |\psi(t)|_{w(U)}^2 \leq C_3^2(||\psi_0||_2^2 + |\psi_0|_{w(U)}^2) \leq C_3^2\epsilon_2^2 \leq \tfrac{\epsilon_3^2}{4}, \end{gathered}$$

for $t \in [0, 2T_0]$.
Using the same method, Local existence and A priori estimate hold for $t \in [0, nT_0]$, $\;n \in \mathbb{N}$. It

means a global solution $\psi \in X(0,\infty)$ exist.

The remain thing is to show ψ_ξ tends to 0 in the L^∞ norm as $t \to \infty$,

$$sup_{\xi \in R} |\psi_\xi(\xi,t)| \to 0 \text{ , as } t \to \infty. \tag{5.0.16}$$

In fact, this is the same method that of Theorem 4.0.2. Using the Gagliardo-Nirenburg interpolation inequality, we get

$$\sup_{\xi \in R} |\psi_\xi(t)| \leq ||\psi_{\xi\xi}(t)||^{1/2} ||\psi_\xi||^{1/2}. \tag{5.0.17}$$

Since

$$|\psi(t)|^2_{w(U)} + \int_0^t \mu |\psi_\xi(\tau)|^2_{w(U)} d\tau \leq C|\psi_0|^2_{w(U)}$$
$$||\psi_\xi(t)||^2 + \int_0^t \mu ||\psi_{\xi\xi}(\tau)||^2 d\tau \leq C||\psi_\xi(0)||^2 + c\int_0^t ||\psi_\xi(\tau)||^2 d\tau \leq C||\psi_0||_1^2$$
$$||\psi_{\xi\xi}(t)||^2 + \int_0^t \mu ||\psi_{\xi\xi\xi}(\tau)||^2 d\tau \leq C||\psi_{\xi\xi}(0)||^2 + c\int_0^t ||\psi_\xi(\tau)||^2 + ||\psi_{\xi\xi}(\tau)||^2 d\tau \leq C||\psi_0||_2^2$$

are done, we know $||\psi_{\xi\xi}(t)||$ is uniformly bounded for $t \geq 0$.

Since $||\psi_\xi(t)||^2$ is integrable over t and $\frac{d}{dt}||\psi_\xi(t)||^2$ is also integrable over t and satisfies

$$\int_0^t \left(\frac{d}{dt}||\psi_\xi(\tau)||^2\right) d\tau = ||\psi_\xi(t)||^2 - ||\psi_\xi(0)||^2 \leq C||\psi_\xi(0)||^2,$$

we know $||\psi_\xi(t)||^2$ is lipschitz continuous. Therefore $||\psi_\xi(t)||^2$ is uniformly continuous. It means

$$||\psi_\xi(t)|| \to 0 \text{ , as } t \to \infty. \tag{5.0.18}$$

So we get,

$$\sup_{\xi \in R} |\psi_\xi(t)| \leq ||\psi_{\xi\xi}(t)||^{1/2} ||\psi_\xi||^{1/2} \to 0, \text{ as } t \to \infty \tag{5.0.19}$$

Therefore, Theorem 5.0.2 is done. □

6

Asymptotic stability of travelling wave for non convex flux case

In this chapter, we check the Asymptotic Stability for non convex flux case, $f'(u_+) < s < f'(u_-)$.

Theorem 6.0.1. *(A priori estimate) For the non convex flux case, the solution $\psi(t)$ in Theorem 5.0.2 satisfies*

$$(1+t)^\gamma ||\psi(t)||_2^2 + \int_0^t (1+\tau)^\gamma ||\psi_\xi(\tau)||_2^2 d\tau \le C(||\psi_0||_2^2 + |\psi_0|_\alpha^2).$$

for any γ , $0 \le \gamma \le \alpha$ and for $0 \le t < T$

Proof. Since $\mu U_\xi = h(U) < 0$ and $U \in (u_+, u_-)$, $U(\xi)$ is a decreasing function in $\xi_* \in R$. It means there exist a unique ξ_* such that it satisfies

$$U(\xi_*) = \tfrac{u_+ + u_-}{2}.$$

Multiplying (3.0.5) by $2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta \psi$, we get

$$2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w\psi\psi_t + 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta \psi\psi_\xi w h'(U) - 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w\psi\mu\psi_{\xi\xi} \tag{6.0.1}$$

$$= 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w\psi F.$$

It's equivalent to the following

$$\begin{aligned} &2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w\psi\psi_t + 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta \psi\psi_\xi w h'(U) - 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w\psi\mu\psi_{\xi\xi} \\ &+(2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta \psi\psi_\xi w' h(U) - 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta \psi\psi_\xi w' h(U)) = 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w\psi F. \end{aligned}$$

Using the product rule, we get

$$2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi\psi_t+2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\psi\psi_\xi(wh)'-2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi\mu\psi_{\xi\xi}$$
$$-2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\psi\psi_\xi w'h(U)=2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi F.$$

Using $h(U)=\mu U_\xi$, we get

$$2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi\psi_t+2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\psi\psi_\xi(wh)'-2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi\mu\psi_{\xi\xi}$$
$$-2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\psi\psi_\xi w'\mu U_\xi=2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi F.$$

Using the product rule, we get

$$\{(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi^2\}_t-\gamma(1+t)^{\gamma-1}\langle\xi-\xi_*\rangle^\beta w\psi^2$$
$$+\{(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta(wh)'\psi^2\}_\xi-(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta(wh)''(U)U_\xi\psi^2-(1+t)^\gamma\langle\xi-\xi_*\rangle^{\beta-2}\beta(\xi-\xi_*)(wh)'\psi^2$$
$$+\{-2\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi\psi_\xi\}_\xi+2\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi_\xi^2+2\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^{\beta-2}\beta(\psi-\psi_*)w\psi\psi_\xi=$$
$$2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi F.$$

It means

$$\{(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi^2\}_t-\gamma(1+t)^{\gamma-1}\langle\xi-\xi_*\rangle^\beta w\psi^2+\{\cdot\}_\xi-(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta(wh)''(U)U_\xi\psi^2$$
$$-(1+t)^\gamma\langle\xi-\xi_*\rangle^{\beta-2}\beta(\xi-\xi_*)w\psi^2+2\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi_\xi^2+2\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^{\beta-2}\beta(\psi-\psi_*)w\psi\psi_\xi=$$
$$2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w\psi F,$$

where

$$A_\beta(\xi)=-\langle\xi-\xi_*\rangle U_\xi(h(U))''-\beta(\xi-\xi_*)\langle\xi-\xi_*\rangle^{-1}(w(U)h(U))'$$
$$=-2\langle\xi-\xi_*\rangle U_\xi-2\beta(\xi-\xi_*)\langle\xi-\xi_*\rangle^{-1}(U-U(\xi_*)).$$

Lemma 6.0.1. *Let α be a given positive number. For $\beta\in[0,\alpha]$, there is a positive number c_0, independent of β such that*

$$A_\beta(\xi)\geq c_0\beta\ ,\ \forall\xi\in R$$

Proof. In this lemma, we prove $A_\beta(\xi)$ has bounded below value.
See $(w(U)h(U))'=-2(U-U(\xi_*))$, then $(w(U)h(U))'|_{\xi=\xi_*}=0$ and $(w(U)h(U))''=-2U(\xi)>0$. So, $(w(U)h(U))'\to u_\pm-u_\mp$ as $\xi\to\pm\infty$. It means for any $\delta>0$,

$$-\beta(\xi-\xi_*)<\xi-\xi_*>^{-1}(w(U)h(U))'\geq c(\delta)\ ,\text{ where }\delta\leq|\xi-\xi_*|$$

next, for some $\delta_0\geq|\xi-\xi_*|$,

$$-<\xi-\xi_*>U_\xi(w(U)h(U))''=-2<\xi-\xi_*>U_\xi=-2<\xi-\xi_*>\tfrac{h(U(\xi))}{\mu}\geq\tfrac{h(U(\xi_*))}{\mu}.$$

So, we know

$$A_\beta(\xi)=-<\xi-\xi_*>U_\xi(w(U)h(U))''-\beta(\xi-\xi_*)<\xi-\xi_*>^{-1}(w(U)h(U))'$$
$$\geq\tfrac{h(U(\xi_*))}{\mu}+c(\delta)\geq c_0\beta\ ,\text{ where }c_0=\min(c(\delta_0),\tfrac{-h(U(\xi_*))}{\mu\alpha})$$

□

Integrating over $[0,t]\times R$ and note $C^{-1}\le w(U)\le C$, we have

$$\begin{aligned}&(1+t)^\gamma|\psi|_\beta^2+\beta\int_0^t(1+\tau)^\gamma|\psi(\tau)|_{\beta-1}^2d\tau+\int_0^t(1+\tau)^\gamma|\psi_\xi(\tau)|_\beta^2d\tau\\ &\le C\{|\psi_0|_\beta^2+\gamma\int_0^t(1+\tau)^{\gamma-1}|\psi(\tau)|_\beta^2d\tau+\beta\int_0^t(1+\tau)^\gamma\int_R\langle\xi-\xi_*\rangle^{\beta-1}|\psi\psi_\xi|d\xi d\tau\\ &+\int_0^t(1+\tau)^\gamma\int_R\langle\xi-\xi_*\rangle^\beta|\psi||F|d\xi d\tau\}\end{aligned}$$

Since $|\psi|\le N(t)$, $|F|\le C\psi_\xi^2$ and

$$C\beta\langle\xi-\xi_*\rangle^{\beta-1}|\psi\psi_\xi|\le\frac{\beta}{2}\langle\xi-\xi_*\rangle^{\beta-1}\psi^2+\frac{C^2\beta}{2}\langle\xi-\xi_*\rangle^{\beta-1}\psi_\xi^2,$$

and for some fixed $R>0$,

$$\int_R\frac{C^2\beta}{2}\langle\xi-\xi_*\rangle^{\beta-1}\psi_\xi^2d\xi=\int_{|\xi-\xi_*|>R}+\int_{|\xi-\xi_*|\le R}\le\frac12|\psi_\xi|_\beta^2+\beta C_R||\psi_\xi||^2,$$

we get

$$\begin{aligned}&(1+t)^\gamma|\psi|_\beta^2+\beta\int_0^t(1+\tau)^\gamma|\psi(\tau)|_{\beta-1}^2d\tau+\int_0^t(1+\tau)^\gamma|\psi_\xi(\tau)|_\beta^2d\tau\\ &\le C\{|\psi_0|_\beta^2+\gamma\int_0^t(1+\tau)^{\gamma-1}|\psi(\tau)|_\beta^2d\tau+\frac1C\int_0^t(1+\tau)^\gamma(\frac12|\psi_\xi|_\beta^2+\beta C_R||\psi_\xi||^2)d\tau\\ &+\int_0^t(1+\tau)^\gamma\int_R\langle\xi-\xi_*\rangle^\beta N(\tau)C\psi_\xi^2d\xi d\tau\}.\end{aligned}$$

It becomes

$$\begin{aligned}&(1+t)^\gamma|\psi|_\beta^2+\int_0^t\{\frac{\beta}{2}(1+\tau)^\gamma|\psi(\tau)|_{\beta-1}^2+(\frac12-CN(\tau))(1+\tau)^\gamma|\psi_\xi(\tau)|_\beta^2\}d\tau\\ &\le C\{|\psi_0|_\beta^2+\gamma\int_0^t(1+\tau)^{\gamma-1}|\psi(\tau)|_\beta^2d\tau+\beta\int_0^t||\psi_\xi(\tau)||^2d\tau\}\end{aligned}$$

Thus we get the following

Lemma 6.0.2. *There is a positive constant ϵ_5 such that if $N(T)<\epsilon_5$, then for $t\in[0,T]$,*

$$\begin{aligned}&(1+t)^\gamma|\psi|_\beta^2+\int_0^t\{\beta(1+\tau)^\gamma|\psi(\tau)|_{\beta-1}^2+(1+\tau)^\gamma|\psi_\xi(\tau)|_\beta^2\}d\tau\\ &\le C\{|\psi_0|_\beta^2+\gamma\int_0^t(1+\tau)^{\gamma-1}|\psi(\tau)|_\beta^2d\tau+\beta\int_0^t||\psi_\xi(\tau)||^2d\tau\}\end{aligned}$$

Using the induction to Lemma 6.0.2, we get a new lemma

Lemma 6.0.3. *For a non negative integer $k\le[\alpha]$,*

$$\begin{aligned}&(1+t)^k|\psi|_{\alpha-k}^2+\int_0^t\{(\alpha-k)(1+\tau)^k|\psi(\tau)|_{\alpha-k-1}^2+(1+\tau)^k|\psi_\xi(\tau)|_{\alpha-k}^2\}d\tau\\ &\le C|\psi_0|_\alpha^2\end{aligned}$$

Consequently, if α is an integer, then the following estimate holds for $0\le\gamma\le\alpha$,

$$(1+t)^\gamma||\psi||^2+\int_0^t(1+\tau)^\gamma||\psi_\xi(\tau)||^2d\tau\le C|\psi_0|_\alpha^2 \tag{6.0.2}$$

Proof. Step 1. letting $\beta=\alpha$ and $\gamma=0$ in Lemma 6.0.2, and $k=0$ in Lemma 6.0.3,

$$|\psi|_\alpha^2 + \int_0^t \{\alpha|\psi(\tau)|_{\alpha-1}^2 + |\psi_\xi(\tau)|_\alpha^2\} d\tau \leq C\{|\psi_0|_\alpha^2 + \alpha \int_0^t ||\psi_\xi(\tau)||^2 d\tau\}$$
$$|\psi|_\alpha^2 + \int_0^t \{(\alpha)|\psi(\tau)|_{\alpha-1}^2 + |\psi_\xi(\tau)|_\alpha^2\} d\tau \leq C|\psi_0|_\alpha^2$$

It means we prove (6.0.2) for $\gamma = 0$ such that it's done for $\alpha < 1$.

Step 2. letting $\beta = 0$, $\gamma = 1$ in Lemma 6.0.2 and $k = 0$ in Lemma 6.0.3,

$$(1+t)^1||\psi||^2 + \int_0^t \{(1+\tau)^1||\psi_\xi(\tau)||^2\} d\tau \leq C\{||\psi_0||^2 + \int_0^t ||\psi(\tau)||^2 d\tau\}$$
$$|\psi|_\alpha^2 + \int_0^t \{(\alpha)|\psi(\tau)|_{\alpha-1}^2 + |\psi_\xi(\tau)|_\alpha^2\} d\tau \leq C|\psi_0|_\alpha^2$$

then we prove (6.0.2) for $\gamma = 1$ where $1 \leq \alpha < 2$.

letting $\beta = \alpha - 1$, $\gamma = 1$ in Lemma 6.0.2, $k = 0$ in Lemma 6.0.3, and $\gamma = 1$ in (6.0.2)

$$(1+t)^1|\psi|_{\alpha-1}^2 + \int_0^t \{(\alpha-1)(1+\tau)^1|\psi(\tau)|_{\alpha-2}^2 + (1+\tau)^1|\psi_\xi(\tau)|_{\alpha-1}^2\} d\tau$$
$$\leq C\{|\psi_0|_{\alpha-1}^2 + \int_0^t |\psi(\tau)|_{\alpha-1}^2 d\tau + (\alpha-1)\int_0^t ||\psi_\xi(\tau)||^2 d\tau\}$$
$$|\psi|_\alpha^2 + \int_0^t \{(\alpha)|\psi(\tau)|_{\alpha-1}^2 + |\psi_\xi(\tau)|_\alpha^2\} d\tau \leq C|\psi_0|_\alpha^2$$
$$(1+t)^1||\psi||^2 + \int_0^t (1+\tau)^1||\psi_\xi(\tau)||^2 d\tau \leq C|\psi_0|_\alpha^2$$

then we have the estimate

$$(1+t)^k|\psi|_{\alpha-k}^2 + \int_0^t \{(\alpha-k)(1+\tau)^k|\psi(\tau)|_{\alpha-k-1}^2 + (1+\tau)^k|\psi_\xi(\tau)|_{\alpha-k}^2\} d\tau$$
$$\leq C|\psi_0|_\alpha^2.$$

is done for $k = 1$

Step 3. letting $\beta = 0$, $\gamma = 2$ in Lemma 6.0.2 and $k = 1$ in Lemma 6.0.3,

$$(1+t)^2||\psi||^2 + \int_0^t \{(1+\tau)^2||\psi_\xi(\tau)||^2\} d\tau \leq C\{||\psi_0||^2 + 2\int_0^t (1+\tau)^1||\psi(\tau)||^2 d\tau\}$$
$$(1+t)^1|\psi|_{\alpha-1}^2 + \int_0^t \{(\alpha-1)(1+\tau)^1|\psi(\tau)|_{\alpha-2}^2 + (1+\tau)^1|\psi_\xi(\tau)|_{\alpha-1}^2\} d\tau \leq C|\psi_0|_\alpha^2$$

then we prove (6.0.2) for $\gamma = 2$ where $2 \leq \alpha < 3$.

letting $\beta = \alpha - 2$, $\gamma = 2$ in Lemma 6.0.2, $k = 1$ in Lemma 6.0.3, and $\gamma = 2$ in (6.0.2),

$$(1+t)^2|\psi|_{\alpha-2}^2 + \int_0^t \{(\alpha-2)(1+\tau)^2|\psi(\tau)|_{\alpha-3}^2 + (1+\tau)^2|\psi_\xi(\tau)|_{\alpha-2}^2\} d\tau$$
$$\leq C\{|\psi_0|_{\alpha-2}^2 + 2\int_0^t (1+\tau)^1|\psi(\tau)|_{\alpha-2}^2 d\tau + (\alpha-2)\int_0^t ||\psi_\xi(\tau)||^2 d\tau\}$$
$$(1+t)^1|\psi|_{\alpha-1}^2 + \int_0^t \{(\alpha-1)(1+\tau)^1|\psi(\tau)|_{\alpha-2}^2 + (1+\tau)^1|\psi_\xi(\tau)|_{\alpha-1}^2\} d\tau \leq C|\psi_0|_\alpha^2$$
$$(1+t)^2||\psi||^2 + \int_0^t (1+\tau)^2||\psi_\xi(\tau)||^2 d\tau \leq C|\psi_0|_\alpha^2$$

then we have the estimate

$$(1+t)^k|\psi|_{\alpha-k}^2 + \int_0^t \{(\alpha-k)(1+\tau)^k|\psi(\tau)|_{\alpha-k-1}^2 + (1+\tau)^k|\psi_\xi(\tau)|_{\alpha-k}^2\} d\tau$$
$$\leq C|\psi_0|_\alpha^2.$$

is done for $k = 2$

Repeating these steps, the term k in Lemma 6.0.3 and γ in (6.0.2) increase until $k = \gamma = \alpha$. It means Lemma 6.0.3 is done. $\square$

Lemma 6.0.4. *For any α, there exists a positive constant ϵ_6 such that if $N(T) \leq \epsilon_6$, then we get*

$$(1+t)^{\gamma}||\psi_{\xi}(t)||^2 + \int_0^t (1+\tau)^{\gamma}||\psi_{\xi\xi}(\tau)||^2 d\tau \leq C(||(\psi_0)_{\xi}||^2 + |\psi_0|_{\alpha}^2),$$
$$(1+t)^{\gamma}||\psi_{\xi\xi}(t)||^2 + \int_0^t (1+\tau)^{\gamma}||\psi_{\xi\xi\xi}(\tau)||^2 d\tau \leq C(||(\psi_0)_{\xi\xi}||^2 + |\psi_0|_{\alpha}^2).$$

Proof. For the ψ_{ξ} estimate, applying ∂_{ξ} to (3.0.5) and multiplying $2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}$, we get

$$2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}\psi_{\xi t} + 2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}^2(h')_{\xi} + 2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}\psi_{\xi\xi}h'$$
$$-2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}\mu\psi_{\xi\xi\xi} = 2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}F_{\xi}$$

Using the product rule, we get

$$\{(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}^2\}_t - \gamma(1+t)^{\gamma-1}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}^2 + 2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}^2(h')_{\xi}$$
$$+\{(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}h'\psi_{\xi}^2\}_{\xi} - (1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta-2}\beta(\xi - \xi_*)h'\psi_{\xi}^2 - (1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}(h')_{\xi}\psi_{\xi}^2$$
$$-\{2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\mu\psi_{\xi}\psi_{\xi\xi}\}_{\xi} + 2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\mu\psi_{\xi\xi}^2 + 2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta-2}\beta(\xi - \xi_*)\mu\psi_{\xi}\psi_{\xi\xi}$$
$$= 2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}F_{\xi}.$$

Integrating over $(0,t) \times \mathbb{R}$, we get

$$(1+t)^{\gamma}|\psi_{\xi}(t)|_{\beta}^2 - |\psi_{\xi}(0)|_{\beta}^2 + \int_0^t \int_R -\gamma(1+\tau)^{\gamma-1}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}^2 + (1+\tau)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}^2(h')_{\xi} d\xi d\tau$$
$$-\int_0^t \int_R (1+\tau)^{\gamma}\langle \xi - \xi_* \rangle^{\beta-2}\beta(\xi - \xi_*)h'\psi_{\xi}^2 d\xi d\tau + \int_0^t \int_R 2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta-2}\beta(\xi - \xi_*)\mu\psi_{\xi}\psi_{\xi\xi} d\xi d\tau$$
$$+\int_0^t 2(1+\tau)^{\gamma}\mu|\psi_{\xi\xi}(\tau)|_{\beta}^2 d\tau = \int_0^t \int_R 2(1+t)^{\gamma}\langle \xi - \xi_* \rangle^{\beta}\psi_{\xi}F_{\xi}.$$

Since $\langle \xi - \xi_* \rangle^{a} \leq \langle \xi - \xi_* \rangle^{b}$ for $a \leq b$, and $\sup_{\xi}|h'|$ and $\sup_{\xi}|(h')_{\xi}|$ are bounded, we get

$$-\int_0^t \int_R (1+\tau)^{\gamma}\langle \xi - \xi_* \rangle^{\beta-1}\psi_{\xi}^2(h')_{\xi} + (1+\tau)^{\gamma}\langle \xi - \xi_* \rangle^{\beta-2}\beta(\xi - \xi_*)h'\psi_{\xi}^2 d\xi d\tau$$
$$\leq \sup_{\xi}|h'| \int_0^t (1+\tau)^{\gamma}c|\psi_{\xi}(\tau)|_{\beta-1}^2 + \sup_{\xi}|(h')_{\xi}| \int_0^t (1+\tau)^{\gamma}\beta|\psi_{\xi}(\tau)|_{\beta-1}^2 d\tau.$$

By the schwartz inequality, we get

$$2\langle \xi - \xi_* \rangle^{\beta-1}\beta|\psi_{\xi}\psi_{\xi\xi}| \leq \frac{\beta}{2}\langle \xi - \xi_* \rangle^{\beta-1}\psi_{\xi}^2 + 2\beta\langle \xi - \xi_* \rangle^{\beta-1}\psi_{\xi\xi}^2.$$

So, we get

$$(1+t)^{\gamma}|\psi_{\xi}(t)|_{\beta}^2 + \int_0^t \frac{\beta}{2}(1+\tau)^{\gamma}|\psi_{\xi}(\tau)|_{\beta-1}^2 + \frac{1}{2}(1+\tau)^{\gamma}|\psi_{\xi\xi}(\tau)|_{\beta}^2 d\tau$$
$$\leq C\{|\psi_{\xi}(0)|_{\beta}^2 + \int_0^t \gamma(1+\tau)^{\gamma-1}|\psi_{\xi}(\tau)|_{\beta}^2 + (1+\tau)^{\gamma}\beta||\psi_{\xi\xi}(\tau)||^2 d\tau\}$$

By the induction and Lemma 6.0.3, we get

$$(1+t)^{\gamma}||\psi_{\xi}(t)|| + \int_0^t (1+\tau)^{\gamma}||\psi_{\xi\xi}(\tau)|| d\tau \leq C(||\psi_{\xi}(0)||^2 + |\psi(0)|_{\alpha}^2).$$

For the $\psi_{\xi\xi}$ estimate, applying $\partial_{\xi\xi}$ to (3.0.5) and multiplying $2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}$, we get

$$\begin{gathered}2(1+t)^\gamma\psi_{\xi\xi}\psi_{\xi\xi t}+(1+t)^\gamma(2h'(U)\psi_\xi)_{\xi\xi}\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}\\ -2\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}\psi_{\xi\xi\xi\xi}=2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta F_{\xi\xi}\psi_{\xi\xi}.\end{gathered}$$

Since

$$\begin{gathered}2(h'(U)\psi_\xi)_{\xi\xi}\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}=\{h'(U)\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}^2\}_\xi\\ -h'(U)(\langle\xi-\xi_*\rangle^\beta)_\xi\psi_{\xi\xi}^2+3(h'(U))_\xi\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}^2+2(h'(U))_{\xi\xi}\langle\xi-\xi_*\rangle^\beta\psi_\xi\psi_{\xi\xi},\end{gathered}$$

we get

$$\begin{gathered}\{(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}^2\}_t-2\gamma(1+t)^{\gamma-1}\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}^2+\{\}_\xi\\ +(1+t)^\gamma(-h'(U)(\langle\xi-\xi_*\rangle^\beta)_\xi\psi_{\xi\xi}^2+3(h'(U))_\xi\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}^2+2(h'(U))_{\xi\xi}\langle\xi-\xi_*\rangle^\beta\psi_\xi\psi_{\xi\xi})\\ +2\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi\xi}^2+2\mu(1+t)^\gamma(\langle\xi-\xi_*\rangle^\beta)_\xi\psi_{\xi\xi}\psi_{\xi\xi\xi}=2(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta F_{\xi\xi}\psi_{\xi\xi}.\end{gathered}$$

Integrating over $(0,t)\times\mathbb{R}$, we get

$$\begin{gathered}(1+t)^\gamma|\psi_{\xi\xi}(t)|_\beta^2-|\psi_{\xi\xi}(0)|_\beta^2+\int_0^t 2\mu(1+\tau)^\gamma|\psi_{\xi\xi\xi}(\tau)|_\beta^2d\tau\\ =\int_0^t(1+\tau)^\gamma\int_R(h'(U)(\langle\xi-\xi_*\rangle^\beta)_\xi\psi_{\xi\xi}^2-3(h'(U))_\xi\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}^2-2(h'(U))_{\xi\xi}\langle\xi-\xi_*\rangle^\beta\psi_\xi\psi_{\xi\xi})d\xi d\tau\\ +\int_0^t\int_R 2(1+\tau)^\gamma\langle\xi-\xi_*\rangle^\beta F_{\xi\xi}\psi_{\xi\xi}+2\gamma(1+\tau)^{\gamma-1}\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}^2-2\mu(1+\tau)^\gamma(\langle\xi-\xi_*\rangle^\beta)_\xi\psi_{\xi\xi}\psi_{\xi\xi\xi}d\xi d\tau.\end{gathered}$$

By the schwartz inequality, we get

$$2(\langle\xi-\xi_*\rangle^\beta)_\xi\psi_{\xi\xi}\psi_{\xi\xi\xi}\le\frac{\beta}{2}\langle\xi-\xi_*\rangle^{\beta-1}\psi_{\xi\xi}^2+2\beta\langle\xi-\xi_*\rangle^{\beta-1}\psi_{\xi\xi\xi}^2$$

Since $\langle\xi-\xi_*\rangle^a\le\langle\xi-\xi_*\rangle^b$ for $a\le b$, and $\sup_\xi|h'|$, $\sup_\xi|(h')_\xi|$ and $\sup_\xi|(h')_{\xi\xi}|$ are bounded, we get

$$\begin{gathered}\int_R(h'(U)(\langle\xi-\xi_*\rangle^\beta)_\xi\psi_{\xi\xi}^2-3(h'(U))_\xi\langle\xi-\xi_*\rangle^\beta\psi_{\xi\xi}^2-2(h'(U))_{\xi\xi}\langle\xi-\xi_*\rangle^\beta\psi_\xi\psi_{\xi\xi})d\xi\\ \le\sup_\xi|h'|\beta|\psi_{\xi\xi}(t)|_{\beta-1}^2+3c\sup_\xi|(h')_\xi||\psi_{\xi\xi}(t)|_{\beta-1}^2+2\sup_\xi|(h')_{\xi\xi}|(|\psi_\xi(t)|_{\beta-1}^2+|\psi_{\xi\xi}(t)|_{\beta-1}^2).\end{gathered}$$

So, we get

$$\begin{gathered}(1+t)^\gamma|\psi_{\xi\xi}(t)|_\beta^2+\int_0^t(1+\tau)^\gamma|\psi_{\xi\xi\xi}(\tau)|_\beta^2d\tau+\int_0^t(1+\tau)^\gamma\beta|\psi_{\xi\xi}(\tau)|_{\beta-1}^2d\tau+\int_0^t(1+\tau)^\gamma|\psi_\xi(\tau)|_{\beta-1}^2d\tau\\ \le C(|\psi_{\xi\xi}(0)|_\beta^2+\int_0^t\gamma(1+\tau)^{\gamma-1}|\psi_{\xi\xi}(\tau)|_\beta^2d\tau+\beta\int_0^t(1+\tau)^\gamma||\psi_{\xi\xi}(\tau)||^2d\tau)\\ \le C(|\psi_{\xi\xi}(0)|_\beta^2+\int_0^t\gamma(1+\tau)^{\gamma-1}|\psi_{\xi\xi}(\tau)|_\beta^2d\tau+\beta\int_0^t(1+\tau)^\gamma||\psi_{\xi\xi}(\tau)||^2d\tau+\int_0^t(1+\tau)^\gamma||\psi_\xi(\tau)||^2d\tau)\end{gathered}$$

By the induction, we get

$$(1+t)^\gamma||\psi_{\xi\xi}(t)||^2+\int_0^t(1+\tau)^\gamma||\psi_{\xi\xi\xi}(\tau)||^2d\tau\le C(||\psi_{\xi\xi}(0)||^2+|\psi(0)|_\alpha^2).$$

□

Combining Lemma 6.0.3 with Lemma 6.0.4, we get

$$(1+t)^{\gamma}||\psi(t)||_2^2+\int_0^t(1+\tau)^{\gamma}|\psi_\xi(\tau)|_2^2\leq C(||\psi(0)||_2^2+|\psi(0)|_\alpha^2).$$

So, we proved the Theorem 6.0.1. □

Theorem 6.0.1 is true only $0 \leq t \leq T$. So, using the Local existence and Theorem 6.0.1, we make a global solution $\psi \in X(0,\infty)$ exist such that we get asymptotic stability for $f'(u_+) < s < f'(u_-)$.

Theorem 6.0.2. *(Asymptotic stability for $f'(u_+) < s < f'(u_-)$) For the case $f'(u_+) < s < f'(u_-)$, the solution $\psi(t)$ in Theorem 5.0.2 satisfies*

$$(1+t)^{\gamma}||\psi(t)||_2^2+\int_0^t(1+\tau)^{\gamma}||\psi_\xi(\tau)||_2^2d\tau\leq C(||\psi_0||_2^2+|\psi_0|_\alpha^2),$$

for any γ , $0 \leq \gamma \leq \alpha$.

Proof. To prove Theorem 6.0.2, we have to use continuation argument.
Suppose $\epsilon_2 = \min\{\epsilon_3/2, \epsilon_3/2C_3\}$, $C_2 = C_3$.
By the local existence, we know $||\psi_0||_2^2 + |\psi_0|_\alpha^2 \leq \epsilon_2^2 \leq \epsilon_3^2/4$. Because of T_0 is dependent of ϵ_0 , T_0 has the value $T_0(\epsilon_3)$ such that the solution exists on $[0, T_0(\epsilon_3)]$. It means

$$||\psi(t)||_2^2+|\psi(t)|_{w(U)}^2\leq C||\psi(t)||^2\leq 4C(||\psi_0||_2^2+|\psi_0|_\alpha^2)\leq C\epsilon_3^2 \text{ for } t\in[0,T_0].$$

By A priori estimate with $T = T_0(\epsilon_3)$, we know for $0 \leq t \leq T$

$$(1+t)^{\gamma}||\psi(t)||_2^2+\int_0^t(1+\tau)^{\gamma}||\psi_\xi(\tau)||_2^2d\tau\leq C_3^2(||\psi_0||_2^2+|\psi_0|_\alpha^2).$$

So, we get

$$||\psi(t)||_2^2\leq C_3^2(||\psi_0||_2^2+|\psi_0|_\alpha^2)\leq C_3^2\epsilon_2^2\leq\tfrac{\epsilon_3^2}{4}.$$

It means

$$||\psi(T_0)||_2^2\leq C_3^2\epsilon_2^2\leq\tfrac{\epsilon_3^2}{4}.$$

Using above results, we know

$$\psi(\xi,T_0)\in H^2\cap L_{w(U)}^2=H^2 \text{ and } ||\psi(T_0)||_2^2\leq||\psi(T_0)||_2^2+|\psi(T_0)|_\alpha^2\leq\tfrac{\epsilon_3^2}{4}.$$

So, we can apply Local existence again,

$$||\psi(t)||_2^2+|\psi(t)|_{w(U)}^2\leq C||\psi(t)||_2^2\leq 4C(||\psi_0||_2^2+|\psi_0|_{w(U)}^2)\leq C\epsilon_3^2 \text{ for } t\in[T_0,2T_0].$$

Therefore, Local existence holds for $t \in [0, 2T_0]$.
Also, by A priori estimate with $T = 2T_0$, we get

$$(1+t)^{\gamma}||\psi(t)||_2^2+\int_0^t(1+\tau)^{\gamma}||\psi_\xi(\tau)||_2^2d\tau\leq C_3^2(||\psi_0||_2^2+|\psi_0|_\alpha^2)$$
$$\Rightarrow||\psi(t)||_2^2\leq C_3^2(||\psi_0||_2^2+|\psi_0|_\alpha^2)\leq C_3^2\epsilon_2^2\leq\tfrac{\epsilon_3^2}{4},$$

for $t \in [0, 2T_0]$.

Using the same method, Local existence and A priori estimate hold for $t \in [0, nT_0], \quad n \in \mathbb{N}$. It means a global solution $\psi \in X(0, \infty)$ exist.

The remain thing is to show ψ_ξ decay rate in the L^∞ norm,

$$sup_{\xi \in R}|\psi_\xi(\xi, t)| \leq (1+t)^{-\frac{\alpha}{2}}(||u_0 - U||_1^2 + |\psi_0|_\alpha^2). \tag{6.0.3}$$

Using the Gagliardo-Nirenburg interpolation inequality, we get

$$\sup_{\xi \in R}|\psi_\xi(t)| \leq ||\psi_{\xi\xi}(t)||^{1/2}||\psi_\xi||^{1/2}. \tag{6.0.4}$$

Since

$$(1+t)^\alpha||\psi(t)||^2 + \textstyle\int_0^t (1+\tau)^\alpha||\psi_\xi(\tau)||^2 d\tau \leq C||\psi_0||^2,$$
$$(1+t)^\alpha||\psi_\xi(t)||^2 + \textstyle\int_0^t (1+\tau)^\alpha||\psi_{\xi\xi}(\tau)||^2 d\tau \leq C(||\psi_\xi(0)||^2 + |\psi_0|_\alpha^2),$$
$$(1+t)^\alpha||\psi_{\xi\xi}(t)||^2 + \textstyle\int_0^t (1+\tau)^\alpha||\psi_{\xi\xi\xi}(\tau)||^2 d\tau \leq C(||\psi_{\xi\xi}(0)||^2 + |\psi(0)|_\alpha^2,$$

are done, we know $||\psi_{\xi\xi}(t)||$ is uniformly bounded for $t \geq 0$.

Since $||\psi_\xi(t)||^2$ is integrable over t and $\frac{d}{dt}||\psi_\xi(t)||^2$ is also integrable over t and satisfies

$$\textstyle\int_0^t (\frac{d}{dt}||\psi_\xi(\tau)||^2)d\tau = ||\psi_\xi(t)||^2 - ||\psi_\xi(0)||^2 \leq C(||\psi_\xi(0)||^2 + |\psi_0|_\alpha^2).$$

So, we get

$$\begin{aligned} ||\psi_{\xi\xi}(t)||^{1/2}||\psi_\xi(t)||^{1/2} &\leq C(||\psi_\xi(0)|| + |\psi(0)|_\alpha^2)^{1/2}(1+t)^{-\alpha/2}(||\psi_{\xi\xi}(0)|| + |\psi_0|_\alpha^2)^{1/2} \\ &\leq C(1+t)^{-\alpha/2}(||U - u_0||_1^2 + |\psi_0|_\alpha^2). \end{aligned}$$

$\square$

7

Conclusion

In the section 5, we check the stability of travelling wave. Since $h(U)$ is non convex, we had some problem to prove the A priori estimate. In the estimate of ψ, we can't ignore $(h'(U))_\xi$, because it changes its sign. However, we solve it using the suitable weighted function, $w(U)$. Since $w(U)h(U)$ is convex, we change the non convex flux to the convex flux and it proved in the section 4. So, the A priori estimate is proved.
By the continuation argument, we get the following

(i) $f'(u_+) < s < f'(u_-)$ case,
Suppose $u_0 - U$ is integrable and $\psi_0 \in H^2$. Then there exists a constant $\epsilon_1 > 0$ such that if $||\psi_0||_2 < \epsilon_1$, then the viscous scalar conservation with initial value has a unique global solution $u(t,x)$ satisfying

$$u(t,x) - U(x - st) \in C^0([0,\infty); H^1) \cap L^2([0,\infty); H^2)$$

(ii) $f'(u_+) = s < f'(u_-)$ case,
Suppose $u_0 - U$ is integrable and $\psi_0 \in H^2$, there exists a constant $\epsilon_1 > 0$ such that if $||\psi_0||_2 + |\psi_0|_{<\xi>_+} < \epsilon_1$, then the viscous scalar conservation with initial value has a unique global solution $u(t,x)$ satisfying

$$u(t,x) - U(x - st) \in C^0([0,\infty); H^1) \cap L^2([0,\infty); H^2 \cap L_{<\xi>_+})$$
$$\text{where } <\xi>_+ = \begin{cases} \sqrt{1+\xi^2} & (\xi \geq 0) \\ 1 & (\xi < 0) \end{cases}$$

(iii) $f'(u_+) < s = f'(u_-)$ case,
Suppose $u_0 - U$ is integrable and $\psi_0 \in H^2$, there exists a constant $\epsilon_1 > 0$ such that if $||\psi_0||_2 + |\psi_0|_{<\xi>_-} < \epsilon_1$, then the viscous scalar conservation with initial value has a unique global solution $u(t,x)$ satisfying

$$u(t,x) - U(x - st) \in C^0([0,\infty); H^1) \cap L^2([0,\infty); H^2 \cap L_{<\xi>_-})$$
$$\text{where } <\xi>_- = \begin{cases} \sqrt{1+\xi^2} & (\xi < 0) \\ 1 & (\xi \geq 0) \end{cases}$$

Also, since $||\psi_\xi(t)|| \to 0$ as $t \to \infty$, we get

$$\sup_{x\in R} |u(t,x) - U(x - st)| \to 0 \text{ as } t \to \infty.$$

So, the stability of travelling wave is proved.

In the section 6, we check the asymptotic decay rate of $U - u_0$ for $f'(u_+) < s < f'(u_-)$. Since $h(U)$ is non convex, we had some problem to prove the A priori estimate. In the estimate of ψ, because $(h'(U))_\xi$ and $h'(U)$ change their sign, we can't ignore them. However, we solve the estimate of ψ using the suitable weighted function, $w(U)$. Since $w(U)h(U)$ is convex, we change the non convex flux to the convex flux and it's proved in Kawashima and Matsumura [3]. So, the A priori estimate is proved.
By the continuation argument, we get the $U - u_0$ exist in global time and the following

(Asymptotic decay rate for $f'(u_+) < s < f'(u_-)$)
Let u satisfy (i) and $\psi_0 \in L^2$ for some integer $\alpha > 0$. Then it holds

$$\sup_{x\in R} |u(t,x) - U(x - st)| \leq C(1+t)^{-\alpha/2}(||u_0 - U||_1 + |\psi_0|_\alpha)$$

So, we get the asymptotic decay rate $t^{-\alpha/2}$. This decay rate equals to the decay rate of convex function f.

8

Appendix

Theorem 8.0.1. *If $f(t)$ is uniformly continuous on $\mathbb{R}$ and integrable, then $f(t) \to 0$ as $t \to \infty$*

Proof. Suppose there exist a uniformly continuous function f, f is integrable and $f(t) \nrightarrow 0$ as $t \to \infty$. Then f satisfies

$$\limsup_{t\to\infty} |f(t)| = C > 0$$

Since f is uniformly continuous, $f(t)$ is finite for any compact set.

Define the sequence $\{t_n\}$:

$$\{t_n\} = \{t_n; t_n \in [n, n+1], |f(t_n)| = \max_{t\in[n,n+1]} f(t)\}$$

Since $\limsup_{t\to\infty} |f(t)| = C > 0$, there exists N such that $|f(x_n)| \geq \frac{C}{2}$ for all $n \geq N$.

Since f is uniformly continuous, we can fix $\epsilon' < \frac{C}{4}$ and $\delta' = \min\{\frac{1}{2}, \delta\}$

Define the set

$$X_n = t \in [t_{2n}, t_{2n} + \delta'],$$

then $\forall n \geq N$, we get

$$\int |f(t)|dt \geq \int_{\cup X_n} |f|dt \geq \int_{\cup X_n} \frac{C}{4} dt \leq \infty.$$

So, there is a contradiction. $\square$

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